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OPTIMAL PACKINGS OF K_4's INTO A K_n - THE CASE $n \equiv 5 \pmod 6$

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by

A.E. Brouwer

ABSTRACT

In this paper we construct a pairwise group divisible design $GD(4,1,\{2,5^*\};n)$ (i.e. a design with blocks of size 4, groups of size 2 or 5 and exactly one group of size 5, on n points with $\lambda = 1$) for each $n \equiv 5 \pmod 6$ except $n = 11$ or 17 (in which cases such a design does not exist). From these designs optimal packings of K_4's into a K_n are derived for $n \equiv 5 \pmod 6$. This was the last remaining case, the cases $n \equiv 2 \pmod 6$ and $n \not\equiv 2 \pmod 3$ being treated in two earlier papers.

KEY WORDS & PHRASES: *group divisible design, scarce design, packing, constant weight code*

1. INTRODUCTION

Let I_n be a finite set of n elements. For $n \geq k \geq t$ let $D(n,k,t)$ be the largest integer b such that there exist b subsets $B_1,\ldots,B_b$ of I_n, each of k elements, such that every t-element subset of I_n is contained in at most one of them. Our object is to determine $D(n,4,2)$.

This is accomplished for $n \equiv 2 \pmod 6$ in BROUWER & SCHRIJVER [4], for $n \not\equiv 2 \pmod 3$ and $n \neq 19$ in BROUWER [3], for $n = 17$ in BROUWER [2] and for $n \equiv 5 \pmod 6$, $n \neq 17$ in the present paper. Therefore only the value $D(19,4,2)$ remains unknown.

If we define

$$J(n,4,2) = \begin{cases} \lfloor \frac{n}{4} \lfloor \frac{n-1}{3} \rfloor \rfloor - 1 & \text{for } n \equiv 7 \text{ or } 10 \pmod{12} \\ \lfloor \frac{n}{4} \lfloor \frac{n-1}{3} \rfloor \rfloor & \text{otherwise,} \end{cases}$$

then we have the following theorem:

<u>THEOREM</u>. (i) $D(n,4,2) = J(n,4,2)$ iff $n \notin \{8,9,10,11,17,19\}$

(ii) $D(n,4,2) = J(n,4,2) - 2$ for $n \in \{8,11\}$

(iii) $D(n,4,2) = J(n,4,2) - 1$ for $n \in \{9,10,17\}$

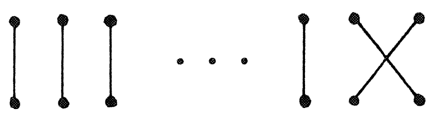
(iv) $J(19,4,2) = 27$, $D(19,4,2) \in \{25,26\}$.

2. OPTIMAL PACKINGS

Taking $n \equiv 5 \pmod 6$ we find $J(n,4,2) = \lfloor \frac{n}{4} \lfloor \frac{n-1}{3} \rfloor \rfloor = \frac{1}{12}(n(n-2)-3)$. Since we always have $D(n,4,2) \leq J(n,4,2)$ [this is the Johnson bound, see e.g. Johnson [6]], an optimal packing of K_4's into a K_n must leave at least

$$\binom{n}{2} - 6 \cdot J(n,4,2) = \frac{n+3}{2}$$

edges uncovered. In the graph formed by the uncovered edges each vertex has valency $\equiv 1 \pmod 3$, hence in the case that $D(n,4,2) = J(n,4,2)$ this graph must look like



(i.e. $(n-5)/2$ disjoint edges and a star on 5 vertices).

For $n = 11$ this is impossible, and it is easily checked that essentially the only way to pack 6 fourtuples into K_{11} is given by the incidence matrix

11110000000
10001110000
10000001110
01001001001
00100100101
00010010011.

For $n = 17$ this is impossible too, but it requires much work to prove this (see [1]). For all other n (with $n \equiv 5 \pmod 6$) we can construct a $GD(4,1,\{2,5^*\};n)$, that is, a group divisible design on n points with blocks of size 4 and one group of size 5, all other groups being of size 2. But this means that the blocks form a packing such that the uncovered edges form $(n-5)/2$ K_2's and one K_5; removing one K_4 from the K_5 leaves the star on 5 vertices just as desired.

Therefore in order to obtain an optimal packing it is sufficient to prove:

<u>THEOREM</u>. *A* $GD(4,1,\{2,5^*\};n)$ *exists iff* $n \equiv 5 \pmod 6$, $n \neq 11,17$.

<u>PROOF</u>. Considering the partition into groups we see that $n \equiv 5 \pmod 2$, and considering the valency of a fixed point that $n - 1 \equiv 1 \pmod 3$. This together with the remarks above proves the 'only if' part. The next section is devoted to the 'if' part.

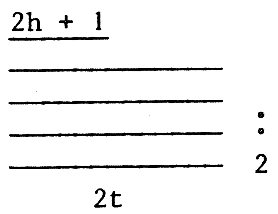
3. THE CLASS $GD(4,1,\{2,5^*\})$.

Let $GD(4,1,\{2,5^*\})$ be the class of all n for which an $GD(4,1,\{2,5^*\}; n)$ exists, and likewise for other designs. Let $V = \{m \mid 6m + 5 \in GD(4,1,\{2,5^*\})\}$. Undefined notations, especially for various types of designs, can be found in HANANI [5] or WILSON [8].

(i) The truncated transversal design.

LEMMA 1. [Truncated Transversal] *If* $h \in V$, $h < t$ *and* $2t \in T(5,1)$ *then* $4t + h \in V$.

PROOF. Removing $2t - (2h+1)$ points from a group of a $T(5,1;2t)$ transversal design, we obtain a $GD(\{4,5\},1,\{2t,(2h+1)^*\};8t+2h+1)$ on some set X. Now construct a $GD(4,1,\{2,5^*\};24t+6h+5)$ on $(X\times I_3) \cup I_2$ as follows:



- construct a $GD(4,1,3;3|B|)$ on the set $B\times I_3$ for each block B of the original design, taking care that it has groups $\{b\} \times I_3$ $(b\in B)$, and take its blocks.
- construct a $GD(4,1,2;6t+2)$ on the set $(G\times I_3) \cup I_2$ for each group G of the original design with $|G| = 2t$, taking care that I_2 is one of its groups, and take all of its blocks and groups except I_2.

[Note that $6t + 2 \in GD(4,1,2)$ iff $t \neq 1$ (see [4]).]

- finally construct a $GD(4,1,\{2,5^*\})$ on the set $(H\times I_3) \cup I_2$, where H is the group of size $2h + 1$, and take all of its blocks and groups. □

Since $2t \in T(5,1)$ for $t \geq 22$ (see HANANI [5]) and $\{0,3,4,5,6\} \subset V$ as we shall see below, this shows that it suffices to prove $m \in V$ for $m \leq 90$, $m \neq 1,2$.

But also $2t \in T(5,1)$ for t even; hence we can do more:

h	t	4t+h
0	2+2s	8+8s
3	4+2s	19+8s
4	6+2s	28+8s
5	6+2s	29+8s
6	8+2s	38+8s
7	8+2s	39+8s
9	10+2s	49+8s
10	12+2s	58+8s

In particular from $\{0,3,4,5,6,7,9,10\} \in V$ it follows that $m \in V$ for $m > 50$.

(ii) The case $n \equiv 5 \pmod{24}$.

LEMMA 2. *Let* $t \neq 1$. *Then* $24t + 5 \in GD(4,1,\{2,5^*\})$.

PROOF. Let $X = (I_{6t+1} \times I_4) \cup \{\infty\}$. Construct a transversal design $T(4,1;6t+1)$ on the set $X \setminus \{\infty\}$ with groups $I_{6t+1} \times \{i\}$, $i \in I_4$, and among the blocks $\{a\} \times I_4$ for some $a \in I_{6t+1}$. Construct for each $i \in I_4$ a group divisible design $GD(4,1,2;6t+2)$ on the set $(I_{6t+1} \times \{i\}) \cup \{\infty\}$ such that $\{(a,i),\infty\}$ is one of its groups. Finally replace the groups $\{(a,i),\infty\}$ and the block $\{a\} \times I_4$ by the group $(\{a\} \times I_4) \cup \{\infty\}$. This yields the required design. □

LEMMA 3. $29 \in GD(4,1,\{2,5^*\})$.

PROOF. Let $X = (Z_3 \times Z_8) \cup \{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5\}$.
Take the groups $\{(0,0),(0,4)\} \bmod (3,8)/2$
and $\{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5\}$,
and the blocks

$\{(0,1),\ (0,3),\ (0,4),\ (1,0)\}$
$\{(1,1),\ (1,3),\ (1,4),\ (2,7)\}$
$\{(0,0),\ (2,0),\ (2,2),\ (2,7)\}$
$\{\ \infty_1\ ,\ (0,0),\ (1,0),\ (2,1)\}$
$\{\ \infty_2\ ,\ (0,0),\ (1,1),\ (2,6)\}$
$\{\ \infty_3\ ,\ (0,0),\ (1,2),\ (2,4)\}$
$\{\ \infty_4\ ,\ (0,0),\ (1,3),\ (2,3)\}$
$\{\ \infty_5\ ,\ (0,0),\ (1,6),\ (2,5)\}$, all mod $(-,8)$.

[Here $\{(0,0),(0,4)\} \bmod (3,8)/2$ means that adding all elements of $\mathbb{Z}_3 \times \mathbb{Z}_8$ to the set $\{(0,0),(0,4)\}$ yields the set of groups twice; it is equivalent with $\{(i,j),(i,j+4)\}$, $i \in \mathbb{Z}_3$, $j = 0,1,2,3$. We shall need this notaiton below.] □

This settles the case $n \equiv 5 \pmod{24}$. In other words: $\forall m$: $4m \in V$.

(iii) Nearly Kirkman Triple systems.

BAKER & WILSON [1] proved that for $n \notin \{1,2,14,17,29\}$ there exists a NKTS $(6n)$ in the notation of KOTZIG & ROSA [7], that is, a resolvable group divisible design RGD $(3,1,2;6n)$ in our notation. Completing this design (i.e. adding a point at infinity for each parallel class of blocks, and the line at infinity as a group) we get a $GD(4,1,\{2,(3n-1)^*\};9n-1)$. Now it follows that if $3n-1 \in GD(4,1,\{2,5^*\})$ then $9n-1 \in GD(4,1,\{2,5^*\})$, provided $n \notin \{1,2,14,17,29\}$.

Assuming (say by inductive hypothesis) that all smaller designs have been constructed this yields $18m-1 \in GD(4,1,\{2,5^*\})$ for $m \geq 4$, $m \neq 7$. But $18 \cdot 7 - 1 = 5.24 + 5$ has been treated in (ii).
Hence:

LEMMA 4. *Let* $m \geq 4$. *Then* $3m-1 \in V$. □

(iv) Multiplying by 5.

LEMMA 5. $t \in V\backslash\{1,3\} \Rightarrow 5t \in V$.

PROOF. Let $X = (I_{2t} \times I_{15}) \cup I_5$. Using a $GD(4,1,3;15)$ on I_{15}, take for each of its blocks B the blocks of a $T(4,1;2t)$ on $I_{2t} \times B$ and for each of its groups G the blocks and groups of a $GD(4,1,\{2,5^*\};6t+5)$ on $(I_{2t} \times G) \cup I_5$ which has I_5 as one of its groups. □

(v) Multiplying by 7.

LEMMA 6. $t \in V\backslash\{2\} \Rightarrow 7t \in V$.

PROOF. Let $X = (I_{3t} \times I_{14}) \cup I_5$. Using a $GD(4,1,2;14)$ on I_{14}, take for each of its groups G the blocks of a $GD(4,1,\{2,5^*\};6t+5)$ on $(I_{3t} \times G) \cup I_5$ which has I_5 as one of its groups. □

(vi) Another way of multiplying by 7.

LEMMA 7. $t \in V \Rightarrow 7t + 5 \in V$.

PROOF. Let $X = I_7 \times I_{6t+5}$. Using a $GD(4,1,\{2,5^*\};6t+5)$ on I_{6t+5}, take for each of its blocks B the blocks of a $T(4,1;7)$ on $I_7 \times B$, for each of its groups G with $|G| = 2$ the blocks and groups of a $GD(4,1,2;14)$ on $I_7 \times G$, and for the group H with $|H| = 5$ the blocks and groups of a $GD(4,1\{2,5^*\};35)$ on $I_7 \times H$. The existence of the latter design (i.e. $5 \in V$) is seen by the following construction:
Let $X = (I_6 \times \mathbb{Z}_5) \cup \{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5\}$.

Take the groups $\{(0,0),(1,0)\}$ mod $(-,5)$

$\{(2,0),(3,0)\}$ mod $(-,5)$

$\{(4,0),(5,0)\}$ mod $(-,5)$

and $\{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5\}$.

Take the blocks

$\{(0,0),(0,1),(2,0),(2,2)\}$,
$\{(0,0),(0,2),(3,3),(3,4)\}$,
$\{(1,0),(1,2),(2,1),(2,2)\}$,
$\{(1,0),(1,1),(3,0),(4,0)\}$,
$\{(1,4),(3,0),(3,2),(4,1)\}$,
$\{(0,0),(3,0),(5,0),(5,2)\}$,
$\{(1,0),(2,3),(5,2),(5,3)\}$,
$\{(0,0),(4,3),(4,4),(5,1)\}$,
$\{(2,0),(4,0),(4,2),(5,1)\}$,
$\{\infty_1,(0,0),(2,3),(4,1)\}$,
$\{\infty_1,(1,3),(3,0),(5,3)\}$,
$\{\infty_2,(0,0),(1,1),(4,2)\}$,
$\{\infty_2,(2,0),(3,1),(5,2)\}$,
$\{\infty_3,(0,0),(1,2),(4,0)\}$,
$\{\infty_3,(2,0),(3,4),(5,3)\}$,
$\{\infty_4,(0,0),(1,3),(5,4)\}$,
$\{\infty_4,(2,0),(3,3),(4,1)\}$,
$\{\infty_5,(0,0),(1,4),(5,3)\}$,
$\{\infty_5,(2,0),(3,2),(4,4)\}$,

all mod $(-,5)$.

This yields the required design. □

(vii) Using a $GD(4,1,\{2,8^*\})$.

LEMMA 8. *If* $6t + 8 \in GD(4,1,\{2,8^*\})$ *and there exists a transversal design* $T(4,1;6t+5)$ *with subdesign* $T(4,1;5)$ *then* $6(4t+3) + 5 \in GD(4,1\{2,5^*\})$, i.e. $4t + 3 \in V$. *In particular* $15 \in V$.

PROOF. Let $X = (I_4 \times I_{6t+5}) \cup I_3$. Take the blocks of a transversal design $T(4,1;6t+5)$ on $I_4 \times I_{6t+5}$ except those of the subdesign say on $I_4 \times A$, where $|A| = 5$. For each $i \in I_4$ take the groups of size 2 and all the blocks of a $GD(4,1,\{2,8^*\};6t+8)$ on $\{i\} \times I_{6t+5} \cup I_3$ which has

$\{i\} \times A \cup I_3$ as its group of size 8.
Finally construct a $GD(4,1\{2,5^*\};23)$ on $(I_4 \times A) \cup I_3$.
[The existence of this design was shown in [4] (v.i).]

Concerning the 'in particular': let $t = 3$, then $6t + 8 = 26 \in GD(4,1,\{2,8^*\})$ as we saw under (iii). The required transversal design was constructed in [4] (v.ii). □

(viii) The case $n \equiv 17 \pmod{24}$

(a) $GD(4,1,2;20)$ with four pairwise disjoint blocks.
I do not know of any $GD(4,1,2;20)$ with a parallel class, i.e. five pairwise disjoint blocks, but the one constructed in [4] (i) has the four disjoint blocks $\{00,01,12,14\}$, $\{02,04,20,21\}$, $\{03,13,34,32\}$, $\{10,11,23,33\}$ (where ij is written instead of (i,j)).

(b) A certain transversal design.
If we take a resolvable design $RB(4,1;12r+4)$, add one point at infinity to some parallel class and remove some other point, we get a $GD(\{4,5\},1,\{3,4^*\};12r+4)$ such that each block of size 5 intersects the group of size 4. By the usual construction (using a $RT(4,1;|B|)$ on $B \times I_4$ for each block B, and a $T(4,1;|G|)$ on $G \times I_4$ for each group G, see e.g. HANANI [5] thm 3.2) we get a $T(4,1;12r+4)$. This transversal design has the following properties:

(α) IF H was the unique group of size 4 of the group divisible design, then this transversal design contains the block $\{h\} \times I_4$ iff $h \in H$.

(β) If A is some fixed group of size 5 of the group divisible design, and $A \cap H = \{a\}$ then the blocks of the transversal design entirely contained within $A \times I_4$ form together with the four blocks $\{b\} \times I_4$, $b \in A \backslash H$, a $T(4,1;5)$.

(c) The construction.
Let $X = (I_{12r+4} \times I_4) \cup \{\infty\}$. We construct a $GD(4,1\{2,5^*\};48r+17)$ on X as follows:
Take the blocks of the transversal design on $I_{12r+4} \times I_4$ con-

structed above except those contained in $A \times I_4$ and the block $\{c\} \times I_4$, where c is some fixed point in $H\backslash A$. Take the blocks and groups of a $GD(4,1,2;20)$ on $A \times I_4$ constructed in such a way that it has $\{b\} \times I_4$, $b \in A\backslash\{a\}$, among its blocks, except for the four blocks mentioned. Take for each $i \in I_4$ the blocks and groups of a $GD(4,1,\{2,5^*\};12r+5)$ on $I_{12r+4} \times \{i\} \cup \{\infty\}$ constructed in such a way that it has $A \times \{i\}$ and $\{(c,i),\infty\}$ among its groups, except for the two groups mentioned. Finally add $\{c\} \times I_4 \cup \{\infty\}$ as a group. This proves that if $2r \in V$ and $r \neq 0$ then $8r + 2 \in V$. The construction under c works as well if we change $12r + 5$ into $6r + 5$, so all we have to do is constructing a suitable transversal design $T(4,1;12r+10)$. But (partially) completing a $RB(4,1;12r+4)$ with 7 points at infinity (which is possible as soon as $4r+1 \geq 7$, i.e. $r \geq 2$) and removing some other point, we get a $GD(\{4,5,7^*\}, 1, \{3,4\}; 12r+10)$.

The $T(4,1;12r+10)$ based on this group divisible design has the properties:

(α) it contains the block $\{b\} \times I_4$ iff $b \in G$ for some group G of size 4.

(β) If A is some fixed group of size 5, then the blocks of the transversal design contained entirely within $A \times I_4$ form together with at most four blocks of the type $\{a\} \times I_4$ a $T(4,1;5)$ on $A \times I_4$.

This time, while carrying out the construction, we have to discard at most four disjoint blocks from the $GD(4,1,2;20)$. Hence

LEMMA 9. $t \in V\backslash\{0,1,3\} \Rightarrow 4t + 2 \in V$. □

Assuming that all smaller designs have been constructed already, this yields all $n \equiv 17 \pmod{24}$ except 17, 41, 65, 89. The design on 17 points does not exist, 89 follows from (iii), and 41 and 65 are given below.

LEMMA 10. $41 \in GD(4,1,\{2,5^*\})$.

PROOF. Let $X = (I_3 \times Z_{12}) \cup \{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5\}$.
Take the groups $\{(i,0),(i,6)\} \bmod (-,12)/2$ $(i \in I_3)$ and $\{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5\}$.

Take the blocks

$\{(0,0),(0,1),(1,0),(1,2)\}$,
$\{(1,0),(1,1),(2,0),(2,2)\}$,
$\{(0,0),(0,4),(0,7),(1,10)\}$,
$\{(1,0),(1,3),(1,7),(2,10)\}$,
$\{(0,0),(0,2),(2,0),(2,5)\}$,
$\{(0,0),(2,4),(2,7),(2,8)\}$,
$\{\infty_1,(0,0),(1,4),(2,9)\}$,
$\{\infty_2,(0,0),(1,5),(2,1)\}$,
$\{\infty_3,(0,0),(1,7),(2,11)\}$,
$\{\infty_4,(0,0),(1,8),(2,2)\}$,
$\{\infty_5,(0,0),(1,9),(2,6)\}$

all mod $(-,12)$. □

LEMMA 11. $65 \in GD(4,1,\{2,5^*\})$.

PROOF. [PDP11] Let $X = Z_3 \times Z_{20} \cup \{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5\}$.
Take the groups $\{(0,0),(0,10)\}$ mod $(3,20)/2$
and $\{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5\}$.
Take the blocks $\{(0,0),(0,1),(0,6),(0,9)\}$,
$\{(0,12),(0,8),(1,5),(2,0)\}$,
$\{(0,14),(0,7),(1,10),(2,0)\}$,
$\{(0,4),(0,6),(1,15),(2,0)\}$ all mod $(3,20)$,

and

$\{(\infty_1,(0,0),(1,0),(2,18)\}$,
$\{(\infty_2,(0,0),(1,19),(2,19)\}$,
$\{(\infty_3,(0,0),(1,18),(2,0)\}$,
$\{(\infty_4,(0,0),(1,1),(2,2)\}$,
$\{(\infty_5,(0,0),(1,2),(2,1)\}$, all mod $(-,20)$. □

Note that this method is generally applicable in the case $n \equiv 5 \pmod{12}$: $X = (Z_3 \times Z_{4t}) \cup I_5$ and the blocks not intersecting I_5 are invariant under $Z_3 \times Z_{4t}$ while the others, though invariant only under Z_{4t}, cover a collection of edges which is invariant under $Z_3 \times Z_{4t}$. [In fact, using a similar solution for $n = 89$ (also found by PDP11), the case $n \equiv 5 \pmod{12}$ can be solved completely without recourse to nearly Kirkman Triple systems.]

(ix) The remaining four cases.

In (ii) and (viii) we proved $t \in V$ for $t \equiv 0 \pmod 2$, $t \neq 2$.

If $t \equiv 1 \pmod 8$ then in (i), using $h = 9$, we saw $t \in V$ for $t \geq 49$;

$t = 9$ has to be done explicitly,

$t = 17$ follows from (iii).

$t = 25$ follows from (iv),

$t = 33$ follows from (vi),

and $t = 41$ follows from (iii).

If $t \equiv 3 \pmod 8$ then in (i), using $h = 3$, we saw $t \in V$ for $t \geq 19$;

$t = 3$ has been done in [4],

and $t = 11$ follows from (iii).

If $t \equiv 5 \pmod 8$ then in (i), using $h = 5$, we saw $t \in V$ for $t \geq 29$;

$t = 5$ has been done in (vi),

$t = 13$ has to be done explicitly,

and $t = 21$ follows from (v).

If $t \equiv 7 \pmod 8$ then in (i), using $h = 7$, we saw $t \in V$ for $t \geq 39$;

$t = 7$ has to be done explicitly,

$t = 15$ follows from (vii),

$t = 23$ follows from (iii),

and $t = 31$ has to be done explicitly.

This leaves $t \in \{7,9,13,31\}$, i.e. $6t + 5 \in \{47,59,83,191\}$.

LEMMA 12. $47 \in GD(4,1,\{2,5^*\})$.

PROOF. Let $X = I_6 \times Z_7$ and construct a $GD(\{3,4\},1,2;42)$ on X such that the triples form 5 Δ-factors (parallel classes). Completion of this design will then yield the required design on 47 points.

Take the groups $\{(i,0),(i+3,0)\}$ mod $(-,7)$, $i = 0,1,2$, the Δ-factors

1. $\{(0,0),(1,5),(5,3)\}$, $\{(2,0),(3,2),(4,6)\}$ mod $(-,7)$
2. $\{(0,0),(2,4),(4,2)\}$, $\{(1,0),(3,5),(5,1)\}$ mod $(-,7)$
3. $\{(0,0),(3,4),(5,1)\}$, $\{(1,0),(2,1),(4,4)\}$ mod $(-,7)$
4. $\{(0,0),(4,1),(5,5)\}$, $\{(1,0),(2,2),(3,1)\}$ mod $(-,7)$
5. $\{(0,0),(4,4),(5,4)\}$, $\{(1,0),(2,3),(3,6)\}$ mod $(-,7)$

and the quadruples

$\{(0,0),(0,1),(1,0),(1,2)\}$, $\{(0,0),(0,2),(2,0),(2,1)\}$,

$\{(0,0),(0,3),(3,1),(3,2)\}$, $\{(0,0),(1,3),(1,4),(2,2)\}$,

$\{(1,0),(1,3),(2,0),(4,1)\}$, $\{(2,0),(2,2),(3,0),(5,1)\}$,

{(2,0),(2,3),(4,0),(5,5)}, {(1,0),(3,0),(3,2),(5,0)},
{(2,0),(3,1),(3,4),(4,2)}, {(0,0),(3,3),(4,3),(4,5)},
{(0,0),(4,6),(4,0),(5,2)}, {(1,0),(3,3),(4,6),(4,2)},
{(0,0),(2,3),(5,6),(5,0)}, {(1,0),(3,4),(5,3),(5,6)}
{(1,0),(4,3),(5,2),(5,4)}
all mod (-,7). □

LEMMA 13. 59 $\in$ GD(4,1,{2,5*}).

PROOF. Let $X = Z_2 \times (Z_3)^3$ and construct a GD({3,4},1,2;54) on X such that the triples form 5 Δ-factors.
Take the groups {(0,0,0,0), (1,0,0,0)} mod (-,3,3,3) and the Δ-factors
1. {(1,0,0,0),(1,2,1,0),(1,1,2,0)} mod (-,3,3,3)/3
   {(0,1,2,0),(0,0,0,1),(0,2,1,2)} mod (-,3,3,3)/3
2. {(0,0,0,0),(0,1,1,1),(0,2,2,2)} mod (2,3,3,3)/3
3-5.[{(0,0,0,0),(1,0,1,0),(0,1,2,1)} mod (2,3,-,3)] mod (-,-,3,-)
and the quadruples
{(0,0,0,0),(0,2,1,0),(1,0,0,1),(1,2,1,2)} mod (-,3,3,3)
and
{(0,1,0,0),(1,2,1,0),(0,0,0,2),(0,2,0,2)},
{(0,0,0,0),(1,2,1,0),(0,0,1,2),(0,0,2,2)},
{(0,1,1,0),(1,2,1,0),(0,1,1,2),(0,2,2,2)}, all mod (2,3,3,3). □

LEMMA 14. 83 $\in$ GD(4,1,{2,5*}).

PROOF. We shall construct a GD({3,4},1,2;60) where the triples form 23 Δ-factors.

(a) Four partitions of Z_{20} each consisting of 5 triples and 5 singletons, such that the triples form the twenty shifts of {0,3,12}, and each point occurs once as a singleton:
1. {0,3,12 },{1,4,13 },{2,5,14 },{6,9,18 },{7,10,19},{8},{11},{15},{16},{17}.
2. {3,6,15 },{4,7,16 },{5,8,17 },{18,1,10},{19,2,11},{0},{ 9},{12},{13},{14}.
3. {8,11,0 },{9,12,1 },{13,16,5},{14,17,6},{15,18,7},{2},{ 3},{ 4},{10},{19}.
4. {10,13,2},{11,14,3},{12,15,4},{16,19,8},{17,0,9 },{1},{ 5},{ 6},{ 7},{18}.

(b) The construction.
Let $X = I_3 \times Z_{20}$. Take the blocks of a RT(3,1;20) and furthermore on each set $\{i\} \times Z_{20}$ the blocks {0,3,12} and {0,1,5,7} (mod 20)

and the groups $\{0,10\}$ (mod 20)/2. This yields a GD($\{3,4\}$,1,2;60). We may suppose that one of the parallel classes of the resolvable transversal design was $\{I_3 \times \{j\} \mid j \in Z_{20}\}$, and by (a) we may partition the union of this parallel class and all 'horizontal' triples into 4 parallel classes. Together with the remaining 19 parallel classes of the transversal design this shows that all triples can be partitioned into 23 Δ-factors. □

LEMMA 15. $191 \in GD(4,1,\{2,5^*\})$.

PROOF. We shall construct a GD($\{3,4\}$,1,2;132) where the triples form 59 Δ-factors.

(a) A 44×44 latin square with 5 increasing diagonals.
A transversal of a latin square is called an increasing diagonal if it is parallel to the main diagonal, and each entry is one more than the one immediately left-above it (here rows, columns and entries are thought of as elements of the cyclic group Z_n).
For instance 021 and 02413 are latin squares where all (3 resp.5) diagonals

021	02413
210	41302
102	30241
	24130
	13024

are increasing. For even orders such latin squares do not exist. However,
0231 has one increasing diagonal.
3102
1320
2013
Forming the direct product with an 11×11 LS with 11 increasing diagonals yields a 44×44 LS with 11 increasing diagonals. (The symbols here are (0,0),(0,1),(0,2),(0,3),(1,0),...,(10,3) in this sequence.)
Even more is true: 0231 and 0213 are mutually orthogonal, showing that there

0231	0213
3102	2031
1320	1302
2013	3120

is a RT (3,1;4) with 1 cyclic parallel class, and by taking the direct product with an 11×11 LS with 11 increasing diagonals (i.e. a cyclic RT(3,1;11)) we get a RT (3,1;44) with 11 cyclic parallel classes.

(b) The construction.

Let $X = I_3 \times Z_{44}$. Take a resolvable transversal design RT (3,1;44) with 5 cyclic parallel classes on X. Use 39 of its 44 parallel classes as they are, leaving 5 cyclic sets $\{(0,a_i),(1,b_i),(2,c_i)\}$ mod 44 $(i = 1,2,3,4,5)$ whose triples will be distributed differently over the remaining 20 Δ-factors we still have to form. Next cover each $\{i\} \times Z_{44}$ $(i \in I_3)$ as follows:

(α) take the matching $\{0,22\}$ (mod 44)/2,

(β) take the quadruples $\{0,4,20,25\}$ (mod 44),

(γ) take the triples $\{0,12,27\}$,$\{0,8,10\}$,$\{0,3,9\}$,$\{0,7,18\}$,$\{0,1,14\}$, all mod 44.

Now all we have to do is to form the remaining 20 Δ-factors. Each cyclic set of triples within $\{i\} \times Z_{44}$ $(i \in I_3)$ together with a cyclic set from the RT (3,1;44) will yield 4 Δ-factors. As follows:

If we have the 'horizontal' triple $\{0,p,q\}$ and the 'vertical' one $\{(0,u_0),(1,u_1),(2,u_2)\}$ then form one Δ-factor by taking on $\{i\} \times Z_{44}$: $\{0,p,q\} + u_i + \lambda_j$ $(0 \le j \le 10)$ where λ is chosen such that the 33 numbers $0 + \lambda j$, $p + \lambda j$, $q + \lambda j$ are all different (and in particular $(\lambda,11) = 1$). This leaves 11 points on each $\{i\} \times Z_{44}$, one in each congruence class mod 11. Since they are shifted the right amount u_i they form 11 blocks from $\{(0,u_0),(1,u_1),(2,u_2)\}$, thus completing the first Δ-factor.

Shifting all blocks by 11, 22, or 33 gives three more.

Remains to show that λ can be chosen suitably.

For $\{0,12,27\}$ choose $\lambda = 1$,

for $\{0,8,10\}$ choose $\lambda = 3$,

and for the other three triples choose $\lambda = 4$. □

This completes the proof of our theorem.

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